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# Relative Chebyshev Centers in Normed Linear Spaces, I*

D. AMIR

*Department of Mathematical Sciences, Tel-Aviv University, Tel-Aviv, Israel*

AND

Z. ZIEGLER

*Department of Mathematics, Technion—Israel Institute of Technology, Haifa, Israel*

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## 0. INTRODUCTION

A Chebyshev center (T-center) of a set in a normed linear space is a single point best approximating the set. The systematic study of T-centers was initiated by Garkavi ([12, 13]).

Our interest lies in investigating the concept of a relative T-center. This concept arises when the elements competing for closeness are restricted to a prescribed family. We divide the discussion into two papers. In the first one, we develop the connection between structural properties of relative centers, convexity properties of the spaces, and the resemblance of the space to a pre-Hilbert space.

In Section 1 we introduce the concept of strict convexity of a space $E$ with respect to a subspace $F$, and prove that the center of every compact set in $E$ with respect to $F$ is at most one point iff $E$ is strictly convex with respect to $F$. We then apply these results to establish that both $C_0(T)$, and the space of continuous functions endowed woth the $L_1$-norm are not strictly convex with respect to any $F$ with $\dim F \geqslant 2$.

Another approach to the structural analysis is via the concept of uniform convexity. We introduce the notion of uniform convexity of $E$ with respect to every direction in $F$ (Uced-$F$). Generalizing work of Day *et al.* [7] and Garkavi, we prove that the center of every bounded $A \subset E$ with respect to $F$ is at most one point iff $E$ is Uced-$F$.

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Section 2 is devoted to the important special case of a relative center with respect to $F$ of a set containing two elements. This is exactly the question of best simultaneous approximation that has been extensively studied in recent years in a less general framework and from a different perspective (e.g., [4, 5, 8, 10]). It is also strongly related to the question of vectorial approximation (see, e.g., [14, 21]). We establish some general results for such centers and relate them to the concept of inner product spaces. Extending results of Rozema and Smith [25] and Garkavi [13], we prove in particular that if the relative centers with respect to all two-dimensional subspaces $F$ of $E$ intersect the line segments $[P(F; 0), P(F; x)]$ then $E$ is an inner product space.

The final section contains characterizations of "homogeneously embedded" subspaces in some important cases. It is shown that the property is quite restrictive. In fact, in $E = C[a, b]$ the sole homogeneously embedded finite-dimensional subspace is span 1. An analysis of the general $C(T)$ and $L_1(T; \mu)$ for a compact $T$, and a $\sigma$-finite $\mu$, is also provided.

In the second part of this work, to be published separately, we develop extensively the $C[0, 1]$ theory of relative centers, establish uniqueness properties, as well as methods for choosing the "best of the best" in cases of nonuniqueness, and discuss continuity questions.

## 1. Relative Centers and Convexity Properties

We start by describing the general set-up. Let $A$ be a bounded set in the metric space $E$, and let $G \subset E$ be an arbitrary set. For $x \in E$, we denote

$$r(x, A) = \inf\{r: A \subset B(x, r)\} \tag{1.1}$$

and define the *relative Chebyshev radius* of $A$ in $G$ by

$$r(G, A) = \inf\{r(x, A): x \in G\}. \tag{1.2}$$

Denote finally the set of relative Chebyshev centers of $A$ in $G$ by $Z(G; A)$, i.e.,

$$Z(G; A) = \{x \in G: r(x, A) = r(G, A)\}. \tag{1.3}$$

Observe that $Z(G; A)$ is the set of centers, in $G$, of balls of minimal radius covering $A$. In another context (cf., e.g., [10]) it is also called the set of best simultaneous approximations to $A$ from $G$.

In the special case where $G = E$, we speak about the (absolute) Chebyshev radius $r(A)$ and the (absolute) Chebyshev center $Z(A)$. If $A = \{y\}$ is a singleton, then $r(x, A) = d(x, y)$, $r(G, A) = d(y, G)$—the distance from $y$

to $G$, and $Z(G; A)$ is the metric projection (or set of best approximations) of $y$ onto $G$, $P(G, y)$.

We will now record some properties of centers.

(a) Clearly, $Z(G; A)$ is closed in $G$, and we have

$$r(G, A) = r(G, \bar{A}), \qquad Z(G; A) = Z(G; \bar{A}).$$

*Assume henceforth that E is a normed linear space.*

(b) We have $r(G, \overline{\text{conv}}\, A) = r(G, A)$ and $Z(G; \overline{\text{conv}}\, A) = Z(G; A)$.

(c) If $G$ is convex, then so is $Z(G; A)$.

(d) If $E$ is a dual space, then $Z(G; A)$ is $w^*$-closed in $G$.

The structure of the center is tied to convexity properties of the spaces $E$ and $G$. We need the following generalization of strict convexity.

DEFINITION 1.1. The space $E$ is said to be *strictly convex with respect to its linear subspace $F$* if its sphere contains no segment parallel to $F$, i.e.,

$$\left\{\| x \| = \| y \| = \left\| \frac{x+y}{2} \right\| = 1, x - y \in F\right\} \Rightarrow x = y. \tag{1.4}$$

We note that $E$ is a strictly convex space iff it is strictly convex with respect to itself. If $E$ is strictly convex with respect to $F$, then it is obviously strictly convex with respect to every $G \subset F$. Furthermore, in this case, each subspace $E_0$, $F \subset E_0 \subset E$, is strictly convex with respect to $F$, and in particular $F$ itself is a strictly convex subspace. A converse relation exists, namely, if $E$ is strictly convex with respect to every one-dimensional $G \subset F$, then $E$ is strictly convex with respect to $F$.

We now relate this concept to the size of centers in the subspace $F$.

LEMMA 1.2. *The following statements are equivalent:*

(i) *$E$ is strictly convex with respect to $F$.*

(ii) *For every compact set $K$, $K \subset E$, the center $Z(F; K)$ is at most a singleton.*

(iii) *For each pair $x, y \in E$, the center $Z(F; \{x, y\})$ is at most a singleton.*

*Proof.* (a) (i) $\Rightarrow$ (ii). We may assume that $r(F, K) = 1$.

If $z, w \in Z(F; K)$, then clearly $(z + w)/2 \in Z(F; K)$. Using the compactness of $K$, it follows that there exists an $x \in K$ such that

$$\left\| x - \frac{z+w}{2} \right\| = 1.$$

Since $r(F, K) = 1$, it follows that $\| x - z \| = \| x - w \| = 1$.

Since $(x - z) - (x - w) = w - z \in F$, the strict convexity of $E$ with respect to $F$ implies that $w = z$.

(b) (ii) $\Rightarrow$ (iii). Self evident.

(c) (iii) $\Rightarrow$ (i). Assume (i) does not hold. Let $x, y$ be such that $\| x \| = \| y \| = \|(x + y)/2 \| = 1$, and $x - y \in F$, $x \neq y$. Then $(x - y)/2$ and $(y - x)/2$ both belong to $Z(F; -(x + y)/2, (x + y)/2$, i.e.,

$$Z\left(F: -\frac{x-y}{2}, \frac{x+y}{2}\right) \text{ is not a singleton.}$$

We recall (see, e.g., [17, p. 109]) the definition of a semi-Chebyshev subset.

DEFINITION. A subset of a normed linear space $E$ is semi-Chebyshev if it contains at most one best approximation to every element of $E$.

Using Lemma 1.2, we immediately derive the following corollary, showing the relation with strict convexity.

COROLLARY 1.3. *If the space $E$ is strictly convex with respect to $F$, then $F$ is semi-Chebyshev in $E$.*

In the special case where $F$ is one-dimensional, this can be strenghtened. In fact, we have

LEMMA 1.4. *If* $\dim F = 1$, *then $E$ is strictly convex with respect to $F$ if and only if $F$ is a Chebyshev set.*

*Proof.* Necessity is covered by the previous corollary. Assume now that $E$ is not strictly convex with respect to $F = [z]$. Let now $x, y$ be such that $x - y = \alpha z \in F$, $x \neq y$, and $1 = \| x \| = \| y \| = \|(x + y)/2 \|$. Then the line $y + t(x - y)$, $-\infty < t < \infty$, is not a Chebyshev set, and therefore $F$ is not a Chebyshev set.

Using Lemmas 1.2 and 1.4, we now have

COROLLARY 1.5. *The space $E$ is strictly convex with respect to $F$ if and only if every one-dimensional subspace of $F$ is a Chebyshev set.*

Further results along these lines are available for special choices of $E$.

ASSERTION 1.6. *The space $C_0(T)$ is not strictly convex with respect to any subspace $F$ with* $\dim F \geqslant 2$.

*Proof.* If $\dim F \geqslant 2$, then some $z \neq 0$ in $F$ has a zero, so that $[z]$ is not a Chebyshev subspace, and the assertion follows by appealing to Corollary 1.5.

ASSERTION 1.7. *Let $\mu$ be any measure. Then $L_1(\mu)$ is not strictly convex with respect to any subspace $F$ with* $\dim F \geqslant 2$. *If $\mu$ is an atomless measure, then $L_1(\mu)$ is not strictly convex with respect to any subspace.*

*Proof.* In the general $L_1(\mu)$ space, the condition for $[v]$ to be a Chebyshev subspace is that

$$\int_A v\, d\mu \neq \int_{A^c} v\, d\mu, \qquad \text{for each measurable } A. \tag{1.5}$$

Suppose $F$ is $n$-dimensional, $n > 2$, and let $v$, $w$ be two linearly independent elements. Let $A$ be fixed. Then there exist $\alpha$, $\beta$ such that

$$\int_A (\alpha v + \beta w)\, d\mu = \int_{A^c} (\alpha v + \beta w)\, d\mu.$$

Hence the one-dimensional subspace spanned by $z = \alpha v + \beta w$ is not a Chebyshev subspace, and $L_1(\mu)$ cannot be convex with respect to $F$ in view of Lemma 1.5.

If $\mu$ is atomless, then the statement is a consequence of the fact that such $L_1(\mu)$ has no finite-dimensional Chebyshev subspaces [23, p. 107].

ASSERTION 1.8. *The space $(C[a, b], \|\cdot\|_1)$ of continuous functions with the $L_1$-norm is not strictly convex with respect to any $F$ with* $\dim F \geqslant 2$.

*Proof.* In order that $[v]$ be a Chebyshev subspace, we must have $\int_a^b v\, dx \neq 0$. As in the proof of the previous assertion, if $F$ has two linearly independent elements, then it has an element $z$ such that $\int_a^b z\, dx = 0$.

Concluding the discussion we provide now an example of a nonstrictly convex $E$ which is strictly convex with respect to $F$, where $\dim F > 1$. Take $F$ as any strictly convex space, $G$ an arbitrary, nonstrictly convex space and $E = F \times G$, where the norm is a strictly convex norm on $R^2$, such as $\|\cdot\|_p$, $1 < p < \infty$.

Another approach to the relation between convexity properties and the nature of the center is via a concept tied to uniform convexity.

DEFINITION 1.9. (a) The space $E$ is said to *uniformly convex with respect to every direction in $F$* (Uced-$F$) if for every $z$, $0 \neq z \in F$ and every $\epsilon > 0$, there exists a $\delta = \delta(z, \epsilon) > 0$ such that

$$\|x\| = \|y\| = 1, \quad x - y = \lambda z, \quad \left\|\frac{x+y}{2}\right\| > 1 - \delta \Rightarrow |\lambda| < \epsilon. \tag{1.6}$$

(b) the space $E$ is *uniformly convex with respect to $F$* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad x - y \in F, \quad \|x - y\| > \epsilon \Rightarrow \left\|\frac{x+y}{2}\right\| > 1 - \delta. \tag{1.7}$$

Note that if $E$ is Uced-$F$ then it is strictly convex with respect to $F$. The case of spaces which are Uced (i.e., where $E = F$) was investigated by Day *et al.* [7]. Garkavi [12] showed that $E$ is Uced iff $Z(A)$ is at most a singleton for every bounded $A \subset E$. A straightforward generalization of Garkavi's proof leads to the following result.

THEOREM 1.10. *The space $E$ is uniformly convex in every direction in $F$ if and only if $Z(F; A)$ is at most a singleton for every bounded $A \subset E$.*

*Proof.* (a) Assume $Z(F; A)$ is not a singleton, and let $y_1, y_2$ be two distinct elements of $Z(F; A)$. Then $y_0 = (y_1 + y_2)/2$ is also in $Z(F; A)$.

Select now a sequence $(x_n) \subset A$ such that $\| y_0 - x_n \| \to r(F; A)$. Then $\| y_i - x_n \| \to r(F; A)$ for $i = 1, 2$. We may assume that $\| y_1 - x_n \| \geqslant \| y_2 - x_n \|$ and take $z_n = y_1 + t_n(y_2 - y_1)$ with $t_n \geqslant 1$ chosen so that $\| z_n - x_n \| = \| y_1 - x_n \|$. Then $u_n = (y_1 - x_n)/\| y_1 - x_n \|$ and $v_n = (z_n - x_n)/\| z_n - x_n \|$ satisfy $\| u_n + v_n \| \to 2$ while $u_n - v_n \in F$ and it does not tend to 0, so that the Uced-$F$ condition is not satisfied.

(b) Conversely, assume $E$ is not Uced-$F$. Then there exists an element $z \in F$ and two sequences $(x_n)$, $(y_n)$ such that

$$\| x_n \| = \| y_n \| = 1, \qquad x_n - y_n = \lambda_n z, \qquad | \lambda_n | \geqslant \lambda > 0 \quad \text{and} \quad \|(x_n + y_n)/2 \| \to 1.$$

Let $u_n = (x_n + y_n)/2$, $A = \{\pm u_n\,;\, n = 1, 2,...\}$. Since $\| u_n \| \to 1$, it follows that $r(F; A) = 1$ and $0 \in Z(F; A)$. However, we have also $\pm\lambda z/2 \in Z(F; A)$ since

$$\left\| u_n \pm \frac{\lambda z}{2} \right\| = \left\| \left(\frac{1}{2} \pm \frac{1}{2\lambda_n}\right) x_n + \left(\frac{1}{2} \pm \frac{\lambda}{2\lambda_n}\right) y_n \right\| \leqslant 1.$$

Hence, $Z(F; A)$ is not a singleton. Q.E.D.

A corresponding result is available for the case where $E$ is uniformly convex with respect to $F$, viz.,

THEOREM 1.11 (cf. [1, 2]). *The space $E$ is uniformly convex with respect to $F$ if, and only if the mapping $A \to Z(F; A)$ is single-valued and uniformly continuous on bounded families of trapezoids (with respect to the Hausdorff metric).*

## 2. Best Simultaneous Approximation of Two Elements

Having discussed the general case and the relation of strict and uniform convexity to the nature of the centers, we now turn our attention to the study of the case where $A = \{x, y\}$, i.e., the best simultaneous approximation of two elements, This question has been extensively studied in recent years in a less general framework and from a different perspective, (e.g., [4, 5, 8, 10, 21]).

We start with a simple observation.

LEMMA 2.1. *Let $E$ be a normed linear space, and let $x, y \in E$, $F$ a subspace of $E$. Suppose that $u \in Z(F; x, y)$. Then exactly one of the following alternatives holds*:

(a) $\| u - x \| = \| u - y \|$,

(b) $u \in Z(F; x)$ *and* $\| u - x \| > \| u - y \|$,

(c) $u \in Z(F; y)$ *and* $\| u - y \| > \| u - x \|$.

*Remark.* Recall that $Z(F; x) = P(F; x)$ is the set of best approximations from $F$ to $x$ (i.e., the metric projection of $x$ into $F$).

*Proof.* Assume (a) does not hold, and that

$$\| u - x \| > \| u - y \| + \epsilon. \tag{2.1}$$

If $u \notin Z(F; x)$, choose $v \in Z(F: x)$, and consider $w = u + \epsilon(v - u)/2 \| v - u \|$. Using the fact that $\| v - x \| < \| u - x \|$, and (2.1), we easily compute

$$\begin{aligned} \| w - x \| &= \left\| \frac{\epsilon(v - x)}{2 \| v - u \|} + \left(1 - \frac{\epsilon}{2 \| v - u \|}\right)(u - x) \right\| \\ &\leqslant \frac{\epsilon \| v - x \|}{2 \| v - u \|} + \left(1 - \frac{\epsilon}{2 \| v - u \|}\right) \| u - x \| < \| u - x \|, \\ \| w - y \| &\leqslant \| u - y \| + (\epsilon/2) < \| u - x \|. \end{aligned}$$

Hence, $r(w; x, y) < \| u - x \| < r(u; x, y)$, contrary to the assumption that $u \in Z(F; x, y)$. Q.E.D.

This lemma indicates a procedure for finding *an* element of $Z(F; x, y)$. Compute first $P(F; x)$ and $P(F; y)$. If one of them is in $Z(F; x, y)$ we are through. If not, consider the line segment

$$[P(F; x), P(F; y)] = \{u: u = \alpha P(F; x) + (1 - \alpha) P(F; y), 0 \leqslant \alpha \leqslant 1\}. \tag{2.2}$$

Choose an $\alpha$ such that $\| u - x \| = \| u - y \|$. If this is in $Z(F; x, y)$, we are through. If not, the problem is reduced to a search in the set $\{v \in F; \| v - x \| = \| v - y \|\}$.

Obviously, the precedure would have been simpler if we could be sure that $Z(F; x, y)$ interesects the segment (2.2). We will show that this property is closely related to $E$ being an inner product space.

We start by noting that Rozema and Smith proved in [25] that if $F$ is a linear subspace of an inner product space $E$, and $A$ is a nonempty bounded subset of $E$, then

$$Z(F; A) \cap \overline{\text{conv}} \left[\bigcup \{P(F; x); x \in A\}\right] \neq \varnothing. \tag{2.3}$$

Specializing to our case, we deduce the following proposition.

PROPOSITION 2.2. *Let $E$ be an inner product space and let $F$ be a linear subspace of $E$. Then*

$$Z(F; x, y) \cap [P(F; x), P(F; y)] \neq \varnothing$$

Garkavi [13] established that if $F = E$, then the validity of $Z(E; A) \cap [\overline{\text{conv}}\, A] \neq \varnothing$ for all nonempty bounded subsets $A$ of $E$ is equivalent to $E$ being an inner product space. We will prove that, in fact, the property that the *relative centers* with respect to all two-dimensional subspaces $F$ intersect the line segments $[P(F; 0), P(F; x)]$ is sufficient to ensure that $E$ is an inner product space.

Recall (see, e.g., [26, p. 93]) that $F$ is a *proximinal subspace* of $E$ if for each element $x \in E$ the set $P_F(x) = P(F; x)$ of best approximants to $x$ from $F$ is not empty. We now introduce a new concept.

DEFINITION 2.3. Let $E$ be a normed linear space and let $F$ be a proximinal subspace. The subspace $F$ is *homogeneously embedded* in $E$ if

$$\{x \in P_F^{-1}(0); y, z \in F; \| y \| = \| z \|\} \Rightarrow \| x - y \| = \| x - z \|. \tag{2.4}$$

Here $P_F$ is the metric projection onto $F$ and $x \in P_F^{-1}(0)$ means that $0 \in P(F; x) = P_F(x)$.

Note that such $F$ is homogeneously embedded in $E$ iff it is a Chebyshev subspace and the intersection of every sphere $S$ in $E$ with $F$ is a sphere in $F$ centered at the best approximation of the center of $S$.

An example of such a subspace is $F$ in the space $(F \oplus G)_p$, $1 \leqslant p < \infty$, where $F$ and $G$ are any normed linear spaces.

*Notation.* Let $E$ be a normed linear space, and let $x \in E$, $r$ a positive number. Then

$$S_E(x; r) = \{u: u \in E, \| u - x \| = r\},$$
$$S_E = S_E(0; 1).$$

THEOREM 2.4. *Let $F$ be a proximinal subspace of the normed linear space $E$. Then the following statements are equivalent*:

(i) *$F$ is homogeneously embedded in $E$.*

(ii) *If $y \in F$, $y \neq 0$ and $x \in P_F^{-1}(y)$, then*

$$\frac{y}{\| y \|} \in P(S_F ; x).$$

(iii) *For every convex subset $A \subset F$ we have*

$$P_A = P_A P_F .$$

(iv) *For every line $L \subset F$, we have $P_L = P_L P_F$.*

(v) *For every segment $I \subset F$, we have $P_I = P_I P_F$.*

*Proof.* (i) $\Rightarrow$ (ii). Suppose $y \in F$, where $F$ is Chebyshev, $y \neq 0$ and $x \in P_F^{-1}(y)$ and that $y/\| y \|$ is not a best approximation to $x$ from $S_F$.

Choosing $z \in P(S_F ; x)$, we have

$$\| x - z \| < \left\| x - \frac{y}{\| y \|} \right\|. \tag{2.5}$$

Let $u$ be the point of intersection of the sphere $\{w; \| w - y \| = | \| y \| - 1 |\}$ and the segment $[y, z]$. There exists such a point, since $\| z \| = 1 \Rightarrow \| y - z \| \geqslant | \| y \| - 1$. We have $\| x - u \| \leqslant \| x - z \|$ since $y \in P_F(x)$; hence, using (2.5) we conclude that the intersection of the sphere (in $E$) $S(x; \| x - y/\| y \| \|)$ with $F$ is not a sphere about the best approximant in $F$ to $x$, namely, $y$. Thus, $F$ is not homogeneously embedded.

(ii) $\Rightarrow$ (i). Assume that $F$ is not homogeneously embedded. Let $x \in P_F^{-1}(0)$, $y, z \in S_F$ and $\epsilon > 0$ be such that

$$\| x - y \| > \| x - z \| + 2\epsilon.$$

Define $x' = (x + \epsilon y)/\| z + \epsilon y \|$, $y' = \epsilon y/\| z + \epsilon y \|$. Since $0 \in P_F(x)$, it follows that $y' \in P_F(x')$. On the other hand,

$$\left\| x' - \frac{y'}{\| y' \|} \right\| = \left\| \frac{x + \epsilon y}{\| z + \epsilon y \|} - y \right\| = \left\| \frac{x - y(\| z + \epsilon y \| - \epsilon)}{\| z + \epsilon y \|} \right\|$$

$$\geqslant \frac{\| x - y \| - 2\epsilon}{\| z + \epsilon y \|} > \frac{\| x - z \|}{\| z + \epsilon y \|} = \left\| x' - \frac{z + y}{\| z + y \|} \right\|$$

so that $y'/\| y' \|$ does not belong to $P(S_F ; x')$. Hence (ii) does not hold.

(iii) $\Rightarrow$ (iv). Trivial.

(iv) $\Rightarrow$ (v). Obvious.

(i) $\Rightarrow$ (iii). Let $A$ be a convex subset of $F$, and let $x \in P_F^{-1}(0)$. Appealing to (i) we conclude that the $\| y \|$-sphere in $F$ touches $A$ at $y$ if and only if the $\| x - y \|$-sphere touches $A$ at $y$. Thus, $P_A(x) = P_A(0) = P_A P_F(x)$.

(v) $\Rightarrow$ (i). The proof of this implication, which is more complicated, may be found, with a different terminology, in [3].

COROLLARY 2.5. *Let $G$ and $F$ be linear subspaces of the normed linear space $E$ and let $G \subset F$. The following inclusion and transitivity relations hold:*

(i) *When $G$ is homogeneously embedded in $E$, it is also homogeneously embedded in $F$.*

(ii) *When $G$ is homogeneously embedded in $F$ and $F$ is homogeneously embedded in $E$, then $G$ is homogeneously embedded in $E$.*

*Proof.* (i) Obvious from the definition.

(ii) Observe first that if $0 \in P_G(x)$ and $y = P_F(x)$, then, by part (iii) of Theorem 2.4, $0 \in P_G(y)$. Since $G$ is homogeneously embedded in $F$, the metric projection of a singleton is a singleton. Hence $0 = P_G(y)$.

Let now $z \in G$. We have

$$S_E(x, \| x - z \|) \cap G = [S_E(x, \| x - z \|)] \cap F \cap G$$
$$= S_F(y, \| y - z \|) \cap G = S_G(0, \| z \|),$$

where the second equality follows from the assumption that $F$ is homogeneously embedded in $E$ and $y = P_F(x)$, whereas the third equality follows from the similar assumptions concerning $G$ in $F$.

We are now ready to relate the concepts of homogeneous embedding and relative centers.

THEOREM 2.6. *Let $F$ be a proximinal subspace of the normed linear space $E$. Consider the following statements:*

(i) *$F$ is homogeneously embedded in $E$.*

(ii) *For all $x, y \in E$ and $u \in P(F; x)$, $v \in P(F; y)$ we have $Z(F; x, y) \cap [u, v] \neq \varnothing$*

(iii) *For all $x \in E$ and $u \in P(F; x)$ we have $Z(F; 0, x) \cap [0, u] \neq \varnothing$.*

*Then* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *If the smooth points of $S_F$ are dense in $S_F$ (this happens, e.g., if $F$ is separable, by Mazur's theorem* [18, *p.* 171]), *then all three statements are equivalent.*

*Proof.* (i) $\Rightarrow$ (ii). Suppose $Z(F; x, y) \cap [u, v] = \varnothing$. Since $u, v \notin Z(F; x, y)$ it follows by Theorem 2.1 that there exists a point $w$, $w \in [u, v]$ such that $\| x - w \| = \| y - w \|$. Since $w \notin Z(F; x, y)$ it now follows that there exists a $z$, $z \in F$ such that $\max(\| x - z)\|, \| y - z \|) < \| x - w \|$. This means that $z$ is interior to the sets $B_E(x, \| x - w \|) \cap F$ and $B_E(y, \| y - w \|) \cap F$, where $B_E(x, a)$ denotes the ball, in $E$, with center at $x$ and radius $a$. However, if $F$

is homogeneously embedded, then $B_E(x, \| x - w \|) \cap F = B_F(u, \| u - w \|)$ and $B_E(y, \| y - w \|) \cap F = B_F(v, \| v - w \|)$ and these two balls have disjoint interiors. Hence our assumption is untenable.

(ii) $\Rightarrow$ (iii). Trivial.

We assume now that the smooth points of $S_F$ are dense in $S_F$, and set to prove that (iii) $\Rightarrow$ (i). Suppose $F$ is not homogeneously embedded in $E$. Then, by Theorem 2.4, there exists a point $x \in P_F^{-1}(0)$ and a segment $I = [y, z] \subset F$ such that $z = P_I(x)$, $y = P_I(0)$ and $\| x - y \| > \| x - z \| + \epsilon$. With no loss of generality we may assume that $\| y \| = 1$ and, in view of the denseness, we may take $y$ to be a smooth point of $S_F$. This implies that $y$ is a smooth point of the sphere $S'$ with radius $\| x - y \|$ about $u = (1 + \| x - y \|) y$. If $z' \in [u, z]$ is such that $\| z' - z \| < \epsilon$, then the segment $[z', y]$ must contain interior points of the $\| x - y \|$-ball in $F$ centered at $u$, since otherwise the segments $[y, z]$ and $[y, z']$ can be extended to supporting hyperplanes of $S_F$. Let $v$ be such an interior point. Then $r(v; (u, x)) < r(y; (u, x))$. Invoking Lemma 2.1, we conclude that

$$Z(F; u, x) \cap [u, P(F; x)] = \varnothing .$$

The translation $x' = x - u$ yields now

$$Z(F; 0, x') \cap [0, P(F; x')] = \varnothing .$$

Hence, (iii) does not hold.

We return now to the relation to inner product spaces. Note that Joichi [19] characterized inner product spaces as the spaces for which every two-dimensional subspace is homogeneously embedded (without using this concept). This property may be deduced from other characterizations of inner product spaces (see [3]). Using this chracterization, we deduce

COROLLARY 2.7. *A normed linear space $E$ is an inner product space if, and only if*

$$Z[F; 0, x] \cap [0, y] \neq \varnothing , \qquad \text{for all } x \in E,\ y \in P(F; x),$$

*where $F$ is any two-dimensional subspace.*

*Remark.* There are various other characterizations of pre-Hilbert spaces It might be interesting to establish a hierarchy between these characterizations. For example, the "Pythagorean" condition

$$x \in P_F^{-1}(0), \qquad y \in F \Rightarrow \| x + y \|^2 = \| x \|^2 + \| y \|^2,$$

is stronger than homogeneous embedding, while the condition

$$x \in P_F^{-1}(0), \qquad y \in F \Rightarrow \| x + y \| = \| x - y \|$$

is weaker, and so is the symmetric orthogonality condition $x \perp F \Rightarrow F \perp x$. We will not pursue here this approach.

## 3. Characterizations of Homogeneously Embedded Subspaces

This section will be devoted to the analysis of homogeneously embedded subspaces. We will see that this property is quite restrictive, and in fact in $E = C[a, b]$ or even $E = C(T)$, where $T$ is any compact metric space, the structure of homogeneously embedded finite-dimensional subspaces is very simple. A complete characterization is then given for homogeneously embedded closed subspaces of $E = L_1(T, \mu)$.

We start with the simplest case of $E = C[a, b]$.

The property of being homogeneously embedded is quite restrictive. In fact, we have

THEOREM 3.1. *The sole homogeneously embedded finite-dimensional subspace of* $C[a, b]$ *is* span 1.

*Proof.* We observe first that if $F =$ span 1 and $x \in P_F^{-1}(0)$, then $\| x \| = \max x(t) = -\min x(t)$. If $\| y \| = \| z \|$, $y, z \in F$, then $y = \pm c$, $z = \pm c$. Since we have $\| x + c \| = \| x - c \| = \| x \| + | c |$, it follows that $F$ is homogeneously embedded.

To prove the converse, it suffices to restrict attention to Chebyshev subspaces. Assume $F$ is an $n$-dimensional Chebyshev subspace of $C[a, b]$ and let $u \in F$ be a function which does not reduce to a constant. Let $\max u(t) - \min u(t) = \delta > 0$, and set $A = \{t; | u(t)| \geqslant \| u \| - \delta/3\}$. Choose a sequence $t_0 < \cdots < t_n$ in $A^C$ and construct now a function $x \in C[a, b]$ satisfying

$$\begin{aligned} \| x \| &= \| u \|, \\ x(t) &= u(t), \qquad \text{for} \quad t \in A, \\ x(t_i) &= (-1)^i \| u \|, \qquad i = 0,..., n. \end{aligned} \tag{3.1}$$

Since $F$ is an $n$-dimensional Chebyshev subspace, and we have $n + 1$ points of alternance for $x - 0$, it follows that $x \in P_F^{-1}(0)$. Observing that $\| x + u \| = 2 \| u \|$, while $\| x - u \| \leqslant 2 \| u \| - (\delta/3)$, it follows that (2.4) is not satisfied, i.e., that $F$ is not homogeneously embedded. Q.E.D.

When we pass to the general $C(T)$ case, the restrictive nature persists, but the proofs are substantially more complicated.

THEOREM 3.2. *Let $T$ be a compact metric space. The only nontrivial homogeneously embedded finite-dimensional subspaces of $C(T)$ are the one-dimensional subspaces spanned by functions $v$, with $|v(t)| \equiv 1$.*

*Proof.* Let $F$ be a proximinal subspace of $C(T)$. Then $x \in P_F^{-1}(0)$ if for every $y \in F$ there are $s, t$ in the *peak set $K_x$ of $x$*, $K_x \equiv \{\omega: |x(\omega)| = \|x\|\}$, such that

$$x(s)\, y(s)\, x(t)\, y(t) \leqslant 0 \tag{3.2}$$

This is analogous to the Kolmogorov condition (see, e.g., [26, Chap. 2]) and is similarly proved. This observation implies that the condition $x \in P_F^{-1}(0)$ is in reality a condition involving only the sets $K_x^+ \equiv \{\omega: x(\omega) = \|x\|\}$, $K_x^- = \{w: x(w) = -\|x\|\}$ and $F$ restricted to these sets.

Suppose now that $|v(t)| = 1$ for every $t \in T$ and let $x \in P_{[v]}^{-1}(0)$. Then we can easily verify, using (3.2), that

$$\|x + v\| = \|x - v\| = \|x\| + \|v\|.$$

Hence, $[v]$ is homogeneously embedded.

Consider next pairs of disjoint closed sets in $T$, $(K^0, K^1)$, such that for every $y \in F$ there exist-$s \in K^i$; $t \in K^j$; $i, j \in \{0, 1\}$, for which

$$(-1)^{i+j}\, y(s)\, y(t) \leqslant 0. \tag{3.3}$$

The family of all such pairs of sets, ordered by

$$(K^0, K^1) \leqslant (L^0, L^1) \qquad \text{iff} \quad K^0 \subset L^0, \quad K^1 \subset L^1,$$

has lower bounds for chains. Indeed, let $(K_\alpha^0, K_\alpha^1)_{\alpha \in A}$ be an infinite chain with $s_\alpha \in K^{i(\alpha)}$, $t_\alpha \in K^{j(\alpha)}$ the corresponding points satisfying (3.3) for a fixed $y \in F$. A compactness argument produces a subnet such that $i(\beta) = i$, $j(\beta) = j$, $s_\beta \to s \in \bigcap K_\alpha^0$, $t_\beta \to t \in \bigcap K_\alpha^1$. We obviously have then $(-1)^{i+j}\, y(s)\, y(t) \leqslant 0$. Thus, $(\bigcap K_\alpha^0, \bigcap K_\alpha^1)$ is a lower bound.

Appealing now to Zorn's lemma, we conclude that there exists a minimal pair $(\mathscr{K}^0, \mathscr{K}^1)$ in $T$. Observe now that the function $x(t)$ defined by

$$x(t) = \frac{d(t, \mathscr{K}^0) - d(t, \mathscr{K}^1)}{d(t, \mathscr{K}^0) + d(t, \mathscr{K}^1)}$$

satisfies $K_x^+ = \mathscr{K}^0$, $K_x^- = \mathscr{K}^1$. Using (3.2) and the definition of $\mathscr{K}^0$, $\mathscr{K}^1$, we conclude that $x \in P_F^{-1}(0)$.

We will now establish the following simple lemma.

LEMMA 3.3. *Let $F$ be homogeneously embedded in $C(T)$, and let $y \in F$,*

$0 \neq x \in P_F^{-1}(0)$. *Then* $K_x \cap K_y \neq \varnothing$ *and there exist* $s, t \in K_x \cap K_y$ *such that* (3.2) *is satisfied.*

*Proof.* If $K_x \cap K_y = \varnothing$, then $\| y \| > \| y \|_{K_x} = \sup\{| y(t)|; t \in K_x\}$, and we may assume that $\| x \| < \frac{1}{2}(\| y \| - \| y \|_{K_x})$

Thus

$$\| y \pm x \| > \tfrac{1}{2}(\| y \| + \| y \|_{K_x}) > \| y \pm x \|_{K_x},$$

and therefore

$$\| y \pm x \| = \| y \pm x \|_{T \setminus K_x}.$$

However, on the set $T \setminus K_x$ we may replace $x$ by $x + h$, where $h$ is supported in a neighborhood of $K_{y+x}$ and satisfies $\| x + h \| < \frac{1}{2}(\| y \| - \| y \|_{K_x})$. We then have $x + h \in P_F^{-1}(0)$ while $\| x + h + y \| \neq \| x + h - y \|$, which is impossible since $F$ is homogeneously embedded.

Hence, $K_x \cap K_y \neq \varnothing$,

$$\| x + y \| = \| x - y \| = \max(\| x + y \|, \| x - y \|) = \| x \| + \| y \|$$

and there exist $s, t \in K_x \cap K_y$ such that

$$y(s)\, y(s)\, x(t)\, y(t) < 0.$$

Returning to the minimal pair $(\mathscr{K}^0, \mathscr{K}^1)$ we denote now $\mathscr{K}^0 \cup \mathscr{K}^1 = \mathscr{K}$. We note that the restriction from $T$ to $\mathscr{K}$ is, by the proof of the lemma, an isometry of $F$ into $C(\mathscr{K})$. By Dugundji's theorem (see [9]) there exists a linear isometry $u: C(\mathscr{K}) \to C(T)$, such that $uf$ is an extension of $f$ for every $f \in C(\mathscr{K})$. By Corollary 2.5(i), $F$ is homogeneously embedded in $uC(\mathscr{K})$. Since the restriction from $T$ to $\mathscr{K}$ is an isometry, it suffices to discuss the case $T = \mathscr{K}$. The possibility of multiplication by $h(t)$, where

$$\begin{aligned} h(t) &= 1, & t &\in \mathscr{K}^0, \\ &= -1, & t &\in \mathscr{K}^1. \end{aligned}$$

which is an autoisometry of $C(\mathscr{K})$, shows that we may assume $\mathscr{K} = \mathscr{K}^0$.

Using the minimality of $\mathscr{K}$ we conclude that for every nonempty open $U$ there exists an element $y \in F$, $\| y \| = 1$, such that $y < 0$ in $\mathscr{K} \setminus U$, and $y$ attains its norm in $U$.

On the other hand, every $y \in F$ must attain both $\| y \|$ and $-\| y \|$ in $\mathscr{K}$. Thus, if $U_1 \cap U_2 = \varnothing$, and $U_i$ are open nonempty sets, the corresponding $y_i$ (whose norm is 1) satisfy $\| y_1 - y_2 \| > 1$. Hence, there are only finitely many such $U_i$'s, and therefore $\mathscr{K}$ is finite. Hence, each $\{t\}$ is an open set, and for each $t \in \mathscr{K}$ there exists $y \in F$, $\| y \| = 1$, with $K_y^+ = \{t\}$ and such that $y(s) < 0$ for $s \neq t$. Let such a $y$ be denoted by $y_s$. Define the set $\sigma(t)$ by

$$\sigma(t) = \{s: y \in F, y(t) = \| y \| \Rightarrow y(s) = -\| y \|\}.$$

Observe that $\sigma(t) \neq \varnothing$ for each $t \in \mathscr{K}$. Indeed, if $y_1, \ldots, y_n \in F$ satisfy $y_i(t) = \| y_i \|$, then necessarily

$$\left\| \sum_{i=1}^{n} y_i \right\| = \sum_{i=1}^{n} y_i(t) = -\sum_{i=1}^{n} y_i(s)$$

for some $s$. Hence, $y_i(s) = -\| y_i \|$, for $i = 1, \ldots, n$, and by compactness $\sigma(t) \neq \varnothing$.

Note that if $s \in \sigma(t)$, then $t \in \sigma(s)$. Indeed, if there is a pair for which the statement is false, there exist $y_s, y_t$ such that $\| y_s - y_t \| = 2 = (y_s - y_t)(s)$ but $y_s - y_t$ does not take the value $-2$. Let now $t_1 \neq t_2$. If there exists a point $s \in \sigma(t_1) \cap \sigma(t_2)$, we choose $y_{t_1}, y_{t_2}$ as above and obtain

$$\| y_{t_1} + y_{t_2} \| = 2 = -(y_{t_1} + y_{t_2})(s)$$

while $y_{t_1} + y_{t_2}$ does not take the value 2. This is impossible, implying $\sigma(t_1) \cap \sigma(t_2) = \varnothing$, and we conclude that the mapping $t \to \sigma(t)$ is an involution of $\mathscr{K}$ onto itself.

For each $y \in F$ and $t \in \mathscr{K}$ we have $y(t) = -y[(\sigma(t)]$. We assume now that $\| y \| = 1$, $y(t) > 0$ and consider $y + y_t$, which also peaks at $t$. We have

$$\begin{aligned} y[\sigma(t)] - y_t(t) &= y[\sigma(t)] + y_t[\sigma(t)] = (y + y_t)[\sigma(t)] \\ &= -(y + y_t)(t) = -y_t - y_t(t). \end{aligned}$$

Thus, $F$ is a subspace of

$$C_\sigma(\mathscr{K}) \equiv \{f \in C(\mathscr{K}); f(t) = -f[\sigma(t)], \text{ for all } t \in \mathscr{K}\}.$$

If $\mathscr{K}$ contains more than two points, we choose $z \in C(\mathscr{K})$ with

$$z(s) = \| z \| = 1, \qquad z[\sigma(s)] = z[\sigma(t)] = 0$$

and we have

$$d(z, F) \geqslant d[z, C_\sigma(K)] = \tfrac{1}{2}.$$

On the other hand, if $y \in F$ peaks at $s$ (i.e., $| y(s) | = \| y \|$) then

$$d(z, \tfrac{1}{2}y) = \tfrac{1}{2} = d(z, \tfrac{1}{2}y + \epsilon u)$$

for every $\epsilon > 0$ small enough, and each $u$ peaking at $t$. We conclude that $F$ is not Chebyshev unless $\mathscr{K}$ consists of two points only $s$, $t$ and $F = \operatorname{span} v$ for some $v \in C(T)$ satisfying $\| v \| = v(s) = -v(t)$.

Since the same argument is valid for every minimal $\mathscr{K}$ for span $v$, it follows that $v$ cannot vanish (if $v(t_0) = 0$, then the singleton $\{t_0\}$ is minimal for $v$), and therefore $| v(t)| = \| v \|$ for every $t \in T$. Q.E.D.

It is well known that $C[a, b]$ has no Chebyshev subspaces of finite codimension $\geqslant 2$. In the general $C(T)$ case, finite codimensional $T$-subspaces exist. However, we have the following theorem.

THEOREM 3.4. *Let $T$ be an infinite compact Hausdorff space. Then $C(T)$ contains no finite codimensional homogeneously embedded subspaces.*

*Proof.* If $F$ is an $n$-codimensional $T$-subspace of $C(T)$, then for every nonzero $\mu \in F^{\perp}$ we have $\operatorname{card}(T \backslash \operatorname{supp} \mu) < n$ [22]. Thus, if $x \in P_F^{-1}(0)$ and $\| x \| = 1$, it follows that $| x(t)| = 1$ except for $m$ points $t_1, ..., t_m$, with $m < n$. We may now assume that $\{t; x(t) = 1\}$ is an infinite open set. Choose in this set $n + 1$ disjoint infinite open sets $U_0, ..., U_n$. Let $u_0, ..., u_n$ be corresponding Urysohn functions, satisfying

$$1 = \| u_i \| \geqslant u_i \geqslant 0 = u_i(T \backslash U_i),$$

$$\tfrac{1}{2} < u_i \qquad \text{on an open infinite set } V_i .$$

Let $\mu_1, ..., \mu_n$ be a basis of $F^{\perp}$ and let $\alpha_0, ..., \alpha_n$ be a nontrivial solution to

$$\sum_{i=0}^{n} \alpha_i \mu_j(u_i) = 0, \qquad j = 0, 1, ..., n.$$

We may assume $\max | \alpha_i | = \max \alpha_i = 1$. Moreover, by modifying a $u_i$, if necessary, we may assume that $| \min \alpha_i | \neq 1$. Then $y = \sum_{i=1}^{n} \alpha_i y_i \in F$ satisfies

$$\| x + y \| = 2 > 1 + | \min \alpha_i | \geqslant \| x - y \|.$$

Hence. $F$ is not homogeneously embedded in $C(T)$. Q.E.D.

We pass next to the $L_1$-case. A complete characterization of the homogeneously embedded closed subspaces is available, viz.,

THEOREM 3.5. *Let $E = L_1(T, \mu)$ where $\mu$ is a $\sigma$-finite measure; then the following statements are equivalent for a closed subspace $F$ of $E$.*

(i) *$F$ is homogeneously embedded in $E$.*

(ii) *$F = \{u \in E: u(t) = 0$ a.e. $(\mu)$ on $A\}$ for some measurable $A \subset T$.*

(iii) *$E = (F \oplus G)_1$, for some subspace $G$.*

*Proof.* (i) $\Rightarrow$ (ii). We observe first that, in $L_1(T, \mu)$, $x \in P_F^{-1}(0)$ iff

$$\left| \int_T \operatorname{sgn} x(t)\, y(t)\, d\mu(t) \right| \leqslant \int_{x^{-1}(0)} | y(t)|\, d\mu(t), \qquad \text{for all } y \in F,$$

(cf. [26, p. 46]). The dependence of this inequality on $x$ is via the $\operatorname{sgn} x$ function only. If $x \in P_F^{-1}(0)$, $y \in F$ and $\mu[(\operatorname{supp} x) \cap (\operatorname{supp} y)] > 0$, we may assume, after multiplying by a scalar, if necessary, that $y(t) - \epsilon > x(t) > 0$ on some set $B$ with $\mu(B) > 0$.

If $\| x + y \| = \| x - y \|$ we may replace $x$ by $x' = x + u$, where $u \in E$ satisfies $u \geqslant 0$, $\| u \| > 0$, $u$ is supported in $B$ and $x + u < y$ on $B$. Then $x' \in P_F^{-1}(0)$, but

$$\begin{aligned} \| x' + y \| &= \| x + y \| + \| u \| \neq \| x + y \| - \| u \| \\ &= \| x - y \| - \| u \| = \| x' - y \|. \end{aligned}$$

This is inconsistent with the assumption that $F$ is homogeneously embedded. Hence $\mu[(\operatorname{supp} x) \cap (\operatorname{supp} y)] = 0$ for every $x \in P_F^{-1}(0)$, $y \in F$. Let now $A$ be a set of maximal measure such that $\mu[A \cap (\operatorname{supp} y)] = 0$ for all $y \in F$. Then each $x \in P_F^{-1}(0)$ is supported in $A$, while each $y \in F$ is supported in $C = T \backslash A$. If $u \in E$ is supported in $C$, then so is also $v = u - P_F u \in P_F^{-1}(0)$. Hence $v = 0$ and $u \in F$. Thus

$$F = \{u \in E \colon \operatorname{supp} u \subset C\} = \{u \in E \colon u(t) = 0 \text{ a.e. } (\mu) \text{ on } A\}.$$

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are immediate. Q.E.D.

We close with a discussion of $C(T)$ endowed with an $L_1$ norm.

THEOREM 3.6. *Let $E = (C(T), \| \cdot \|_{L_1(\mu)})$, where $T$ is a connected locally compact Hausdorff space $T$ and $\mu$ is a Radon measure. Then $E$ has no nontrivial homogeneously embedded subspaces.*

*Proof.* Following the arguments used in the proof of the previous theorem, we find that a homogeneously embedded subspace $F$ must be of the type

$$F = \{u \in E \colon u(t) = 0 \text{ a.e.}(\mu) \text{ on } A\}$$

for some measurable $A \subset T$. If $\mu(A) \neq 0$ and $\mu(T \backslash A) = 0$, such an $F$ cannot be proximinal since the function 1 does not have a best approximation in $F$. Q.E.D.

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